inverses in an automorphism of G this subgroup must be abelian. Hence the following theorem: *The generalized dihedral group whose order is twice an odd number is the only non-abelian group which satisfies the two conditions that it admits an automorphism of order 2 in which only one operator besides the identity corresponds to itself and that this operator is not an automorphism commutator.*

If the non-abelian group G admits an automorphism of order 2 in which only one operator s besides the identity corresponds to itself and is a automorphism commutator then two cases present themselves according as s is invariant under G or does not have this property. The former case is represented by an outer isomorphism of the generalized dihedral or dicyclic groups in which the Sylow subgroups of the order 2^m, $m > 1$, are dihedral or dicyclic whenever $m > 2$. The latter case is represented by the outer isomorphisms of order 2 of the tetrahedral group. It is easy to see that in both cases the Sylow subgroup of order 2^m is always dihedral or dicyclic whenever $m > 2$, since at least one such Sylow subgroup must be invariant under every automorphism of order 2.

[1] G. A. Miller, *Bull. Amer. Math. Soc.*, **6**, 337, 1900.
[2] G. A. Miller, *Trans. Amer. Math. Soc.*, **6**, 60, 1905.

ARITHMETIZED TRIGONOMETRICAL EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND

BY JOHN D. ELDER

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

In a series of papers, Appell[1] discusses the problem of exhibiting a function, $G(z)$, meromorphic and uniform, and satisfying the equations of definition

$$G(z + \pi) = G(z)$$

$$G(z + \pi\tau) = e^{-2miz} G(z)$$
$$m \text{ an integer} \gtrless 0$$

as a sum of simple elements, each having a single singularity in a period parallelogram, and an integral function of z. The two cases, $m > 0$ and $m < 0$, are found to offer essential differences and Appell's method is, in general, of practical use only in the case $m < 0$. Appell's method has been discussed, for $m > 0$, by Mr. M. A. Basoco in a California Institute of Technology dissertation.

Appell defines

$$A_\mu(z, y) = \sum_{n=-\infty}^{n=\infty} e^{2\mu i y n} \, q^{\mu n(n-1)} \cot(z - y - n\pi\tau)$$

$$q = e^{i\pi\tau}$$

and shows that if $F(z)$ is meromorphic and uniform, satisfies

$$F(z + \pi) = F(z)$$

$$F(z + \pi\tau) = e^{2\mu i z} F(z)$$

$$\mu \text{ an integer} > 0$$

and has, in a period parallogram, poles of orders l_i at $z = \alpha_i$, $i = 0, l, \ldots p$ with the corresponding principal parts

$$\sum_{j=1}^{l_i} \frac{R_i^{(j)}}{(z - \alpha_i)^j},$$

then

$$F(z) = \sum_{i=0}^{p} \sum_{j=1}^{l_i} \frac{A_\mu^{(j-1)}(z, \alpha_i)}{(j - 1)!} \, R_i^{(j)},$$

where $A_\mu^{(k)}(z, \alpha_i)$ designates the result of replacing y by α_i in the kth derivative of $A_\mu(z, y)$ with respect to y. It is not feasible to obtain expressions for $A_\mu^{(k)}(z, y)$ by direct differentiation of $A_\mu(z, y)$, but by replacing the cotangents by equivalent expressions in terms of exponentials and expanding, series are obtained which converge uniformly in a strip of the complex plain, whose general derivatives may be taken without difficulty. In this way expressions for $A_\mu^{(k)}[z + a\pi\tau, \, y + a\pi\tau]$, $A_\mu^{(k)}[z, \, y + a\pi\tau]$, $A_\mu^{(k)}[z + (a + \frac{1}{2})\pi\tau, \, y + a\pi\tau]$, and $A_\mu^{(k)}[z + a\pi\tau, \, y + (a + \frac{1}{2})\pi\tau]$ have been obtained which are valid for useful values of z, y, and a.

These expressions have been used to obtain the expansions of all functions of the type

$$F(z) = \vartheta_0^p(z)\vartheta_1^q(z)\vartheta_2^s(z)\vartheta_3^t(z),$$

where p, q, s and t are integers $\gtreqless 0$ for which $-[p + q + t + s]$ equals one, two, or three; and in which the sum of the negative exponents is four or less. Each expansion obtained consists of a finite sum of single trigonometrical terms which show explicitly the singularities of the function expanded which lie on the real axis, and a finite sum of terms, each of which is a singly or doubly infinite series, multiplied in general by theta constants. These series show, on arithmetization, a general uniformity of structure.

The arithmetization is made clear by considering a typical case. One of the series in the expansion of $\vartheta_1^3\vartheta_3^2(z)/\vartheta_0^2\vartheta_2^3(z)$ is

$$2 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{n+r} [2(n+r) - 1]^2 \, q^{n(n+2r-1)} \cos(2r-1)z.$$

On writing $N = n(n + 2r - 1) = b\beta$, the series takes the form

$$2 \sum_N q^N \sum (-1)^{\frac{\beta+b+1}{2}} (\beta + b)^2 \cos(\beta - b)z,$$

where the second summation extends over all pairs, $(b\beta)$, of conjugate divisors of N satisfying $0 < b < \sqrt{N}$, $\beta - b$ odd.

The general result is now stated. Every series in the expansions obtained is of one of the four types

$$\sum q^N \sum (\alpha + \mu a)^l \, {}^{\cos}_{\sin}(\alpha - \mu a)z,$$

$$\sum q^N \sum (\beta + \mu b)^l \, {}^{\cos}_{\sin}(\beta - \mu b)z,$$

$$\sum q^{\frac{N}{4}} \sum (\gamma + \mu c)^l \, {}^{\cos}_{\sin}\frac{\gamma - \mu c}{2}z,$$

$$\sum q^{\frac{N}{4}} \sum (\delta + \mu d)^l \, {}^{\cos}_{\sin}\frac{\delta - \mu d}{2}z,$$

where N equals $a\alpha$, $b\beta$, $c\gamma$, or $d\delta$, respectively, and the pairs (a,α) . . . . satisfy conditions similar to those given in the last paragraph. The sine or cosine is understood according as the function is odd, or is even. Each pole (in the period parallelogram) of order h gives corresponding series in which l takes precisely the values $h - 1$, $h - 3$, ... ending in 0 or 1 according as h is odd or is even. The factors multiplying these series are all simple functions of the corresponding $R_i^{(j)}$.

These expansions, together with the details of the method, are contained in a thesis deposited in the library of the California Institute of Technology. It seems probable that general formulae for the expansion of any function of the type here considered may be obtained, containing as the only undetermined constants the $R_i^{(j)}$. Some progress has been made in this direction and it is hoped to give complete results in a future paper.

[1] *Ann. Sci. l'École Normale Sup.*, Third Series, **1** (p. 135), **2** (p. 2), and **3** (p. 2.)